

Properties of the String Operator in the Eight-Vertex Model.

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Abstract

The construction of creation operators of exact strings in eigenvectors of the eight vertex model at elliptic roots of unity of the crossing parameter which allow the generation of the complete set of degenerate eigenstates is based on the conjecture that the 'naive' string operator vanishes. In this note we present a proof of this conjecture. Furthermore we show that for chains of odd length the string operator is either proportional to the symmetry operator S or vanishes depending on the precise form of the crossing parameter.

1 Introduction

The eight-vertex model of Baxter is a lattice model whose transfer matrix is given by

$$\mathbf{T}_8(v)|_{\mu,\nu} = \text{Tr} W(\mu_1, \nu_1) W(\mu_2, \nu_2) \cdots W(\mu_N, \nu_N) \quad (1)$$

where $\mu_j, \nu_j = \pm 1$ and $W(\mu, \nu)$ is a 2×2 matrix whose non vanishing elements are given as

$$\begin{aligned} W(+1, +1)|_{+1,+1} &= W(-1, -1)|_{-1,-1} = \rho \Theta(2\eta) \Theta(\lambda - \eta) \text{H}(\lambda + \eta) = a(\lambda) \\ W(-1, -1)|_{+1,+1} &= W(+1, +1)|_{-1,-1} = \rho \Theta(2\eta) \text{H}(\lambda - \eta) \Theta(\lambda + \eta) = b(\lambda) \\ W(-1, +1)|_{+1,-1} &= W(+1, -1)|_{-1,+1} = \rho \text{H}(2\eta) \Theta(\lambda - \eta) \Theta(\lambda + \eta) = c(\lambda) \\ W(+1, -1)|_{+1,-1} &= W(-1, +1)|_{-1,+1} = \rho \text{H}(2\eta) \text{H}(\lambda - \eta) \text{H}(\lambda + \eta) = d(\lambda) \end{aligned} \quad (2)$$

where $\text{H}(u)$ and $\Theta(u)$ are Jacobi's Theta functions defined in appendix A. There are several paths leading to its solution. All have been either developed or at least initiated by Baxter in a series of

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famous papers [1]-[4]. The range of validity of the solutions depends on several parameters of the model: It turns out to be essential whether the size N of the lattice is even or odd. Furthermore it is important whether the crossing parameter is generic or restricted to elliptic 'root of unity' values. For details see [1]-[4] and [5]-[8]. We only mention that the TQ equation determines the eigenvalues of transfer matrix T for even N and unrestricted crossing parameter η [2]. There exist well developed methods for the determination of eigenvectors of T for even N and 'root of unity' values of η

$$\eta = 2m_1 K/L + im_2 K'/L \quad (3)$$

See [4],[9],[10]. Concerning the problem to obtain eigenvectors at generic η information is given in [11], footnote 18, in [9] after (5.15) and in [10] on page 497. In the following we restrict the crossing parameter η to elliptic roots of unity and for simplicity to the case $m_2 = 0$. Like in the six vertex model at root of unity the transfer matrix of the eight-vertex model and the Hamiltonian of the related XYZ spin chain have numerous degenerate multiplets of eigenvalues. The symmetry algebra responsible for the degeneracies is well understood in the six vertex model where it is the sl_2 loop algebra [12]. The problem to construct the operators which create the degenerate eigenvectors has been solved in the six vertex model in [13]. This solves simultaneously the problem to construct the current of the sl_2 loop symmetry studied in [12]. The question arises if a similar construction is necessary in the eight vertex model where the eigenvectors depend on free parameters s, t which have no influence on singlet states but affect the degenerate states. For even N eigenvectors of the transfer matrix are given by [9]

$$\psi_k = \sum_{l=0}^{L-1} \exp(2\pi i l k / L) \prod_{m=1}^n B_{l+m, l-m}(\lambda_m, s, t) \Omega_N^{l-n}(s) \quad (4)$$

where $\lambda_1, \dots, \lambda_n$ are Bethe roots. Do eigenvectors obtained by variation of s and t span a complete degenerate subspace? That the answer is no has been shown in [14]. To generate the full degenerate multiplet a new string operator is needed: the creation operator of a complete B -strings is for even N

$$B_l^{L_s, 1}(\lambda_c) = \sum_{j=1}^{L_s} B_{l+1, l-1}(\lambda_1) \cdots \left(\frac{\partial B_{l+j, l-j}}{\partial \eta}(\lambda_j) - \hat{Z}_j \frac{\partial B_{l+j, l-j}}{\partial \lambda}(\lambda_j) \right) \cdots B_{l+L_s, l-L_s}(\lambda_{L_s}) \quad (5)$$

where the arguments λ_k form an exact string of length L_s

$$\lambda_k = \lambda_c - 2(k-1)\eta, \quad k = 1, \dots, L_s \quad (6)$$

λ_c is the string center. This is in addition to s and t a third free parameter.

$\hat{Z}_1(\lambda_c)$ is defined in appendix C.

This problem has also been studied in the framework of the Felder-Varchenko [10] formalism by Deguchi [16].

We note that the string length L_s and the integer L occurring in (3) and (4) are related by the rule [14]

$$L_s = L \quad \text{for odd } L, \quad L_s = L/2 \quad \text{for even } L \quad (7)$$

We now turn to the precise topic of this paper. The construction of the string operator in [14] rests on the conjecture that the 'naive' string operator vanishes :

$$B_s = B_{l+1,l-1}(\lambda_1) \cdots B_{l+L_s,l-L_s}(\lambda_{L_s}) = 0 \quad (8)$$

In the six vertex model the equivalent relation is (see [15])

$$B(\lambda_1) \cdots B(\lambda_{L_s}) = 0 \quad (9)$$

We intend to fill this gap by showing that (8) is satisfied for even N in the eight vertex model. Furthermore we find that for odd N B_s does NOT vanish but is given by a symmetry operator of the model. It is proportional to

$$S = \sigma^3 \otimes \sigma^3 \otimes \cdots \otimes \sigma^3 \quad (10)$$

provided that $2L_s\eta$ is an odd multiple of $2K$. If η is an even multiple of $2K$ B_s vanishes also for odd N .

2 The formalism.

In the following we work in the framework of the algebraic Bethe ansatz for the eight vertex model by Takhtadzhian and Faddeev [9] and use their notation. For convenience some of their basic tools are listed in appendix B.

We shall study the action of operators of type

$$\mathcal{O}_{j+1,l-1}^N = B_{j+1,l-1}^N(\lambda_1) \cdots B_{j+L_s,l-L_s}^N(\lambda_{L_s}) \quad (11)$$

on vectors denoted by Ω^N . The arguments λ_k are defined in (6). The superscript N indicates the size of the one dimensional system. This is needed because our result will be established recursively by relating systems of sizes N and $N - 1$. Ω^N is any element of the set of 2^N independent basis vectors

$$\Omega^N = Z_{l_1} \otimes Z_{l_2} \otimes \cdots \otimes Z_{L_N} \quad (12)$$

where Z_l stands for $X_l(\eta)$ or $Y_l(\eta)$. $X_k(\lambda)$ and $Y_k(\lambda)$ are defined in (B.13) and (B.14). We note that in the algebraic Bethe-Ansatz [9] the eigenvectors of the transfer matrix are obtained by the action of $B_{k,l}$ operators on the system of generating vectors defined in [9] (4.18) and (B.21). The necessity to work in our case with more general basis vectors arises from the intention to prove the operator relation (8). This set of basis vectors is described in detail in (42).

In order to demonstrate the formalism to be used transparently we display the string operator acting

on a basis element of the space of states (in this example for a system of size $N = 4$). What follows is nothing but a much more detailed presentation of the string operator (11) acting on a basis vector (12).

$$\mathcal{O}_{j+1,l-1}\Omega^N = \begin{array}{cccccc} & & Z_{l+2,\sigma_1} & Z_{l+1,\sigma_2} & Z_{l,\sigma_3} & Z_{l-1,\sigma_4} \\ m(\lambda_{L_s})^{-1}\tilde{Y}_{j+L_s}^{\rho_1}(\lambda_{L_s}) & \mathcal{L}_{\delta_1,\rho_1}^{\sigma_1,\rho_2} & \mathcal{L}_{\delta_2,\rho_2}^{\sigma_2,\rho_3} & \mathcal{L}_{\delta_3,\rho_3}^{\sigma_3,\rho_4} & \mathcal{L}_{\delta_4,\rho_4}^{\sigma_4,\rho_5} & Y_{l-L_s,\rho_5}(\lambda_{L_s}) \\ \dots\dots\dots & \dots\dots\dots & \dots\dots\dots & \dots\dots\dots & \dots\dots\dots & \dots\dots\dots \\ m(\lambda_2)^{-1}\tilde{Y}_{j+2}^{\nu_1}(\lambda_2) & \mathcal{L}_{\beta_1,\nu_1}^{\gamma_1,\nu_2} & \mathcal{L}_{\beta_2,\nu_2}^{\gamma_2,\nu_3} & \mathcal{L}_{\beta_3,\nu_3}^{\gamma_3,\nu_4} & \mathcal{L}_{\beta_4,\nu_4}^{\gamma_4,\nu_5} & Y_{l-2,\nu_5}(\lambda_2) \\ m(\lambda_1)^{-1}\tilde{Y}_{j+1}^{\mu_1}(\lambda_1) & \mathcal{L}_{\alpha_1,\mu_1}^{\beta_1,\mu_2} & \mathcal{L}_{\alpha_2,\mu_2}^{\beta_2,\mu_3} & \mathcal{L}_{\alpha_3,\mu_3}^{\beta_3,\mu_4} & \mathcal{L}_{\alpha_4,\mu_4}^{\beta_4,\mu_5} & Y_{l-1,\mu_5}(\lambda_1) \end{array} \quad (13)$$

The \mathcal{L} matrices appearing in the k th row (counted from the bottom) depend on λ_k . One easily recognizes the individual components of (11) in the preceding expression. A single row enclosed between \tilde{Y} and Y represents a B operator written explicitly in terms of local transition matrices as defined in (B.24). The leftmost operator in (11) appears in the lowest row, the basis vector Ω^N in the top row. Expression (13) can be either understood as row operators acting on the upper direct product of Z_l vectors or as column operators which act on the direct product of Y_l vectors appearing in the rightmost column. We prefer the second choice as it takes from the outset into account that the spectral parameters λ_k form a complete string which leads to considerable simplifications. This is an essential part of our approach which enables us to relate different system sizes N recursively. The local transition matrix \mathcal{L} is related to the R matrix by

$$\mathcal{L}_{\alpha,\mu}^{\beta,\nu}(\lambda) = \mathcal{R}_{\alpha,\mu}^{\nu,\beta}(\lambda, \eta) \quad (14)$$

where α, β are quantum indices and μ, ν auxiliary indices in the terminology introduced in [9]. To apply the method of intertwining vectors we rewrite (13) in terms of the R matrix \mathcal{R}

$$\mathcal{O}_{j+1,l-1} = \begin{array}{cccccc} & & Z_{l+2,\sigma_1} & Z_{l+1,\sigma_2} & Z_{l,\sigma_3} & Z_{l-1,\sigma_4} \\ m(\lambda_{L_s})^{-1}\tilde{Y}_{j+L_s}^{\rho_1}(\lambda_{L_s}) & \mathcal{R}_{\delta_1,\rho_1}^{\rho_2,\sigma_1} & \mathcal{R}_{\delta_2,\rho_2}^{\rho_3,\sigma_2} & \mathcal{R}_{\delta_3,\rho_3}^{\rho_4,\sigma_3} & \mathcal{R}_{\delta_4,\rho_4}^{\rho_5,\sigma_4} & Y_{l-L_s,\rho_5}(\lambda_{L_s}) \\ \dots\dots\dots & \dots\dots\dots & \dots\dots\dots & \dots\dots\dots & \dots\dots\dots & \dots\dots\dots \\ m(\lambda_2)^{-1}\tilde{Y}_{j+2}^{\nu_1}(\lambda_2) & \mathcal{R}_{\beta_1,\nu_1}^{\nu_2,\gamma_1} & \mathcal{R}_{\beta_2,\nu_2}^{\nu_3,\gamma_2} & \mathcal{R}_{\beta_3,\nu_3}^{\nu_4,\gamma_3} & \mathcal{R}_{\beta_4,\nu_4}^{\nu_5,\gamma_4} & Y_{l-2,\nu_5}(\lambda_2) \\ m(\lambda_1)^{-1}\tilde{Y}_{j+1}^{\mu_1}(\lambda_1) & \mathcal{R}_{\alpha_1,\mu_1}^{\mu_2,\beta_1} & \mathcal{R}_{\alpha_2,\mu_2}^{\mu_3,\beta_2} & \mathcal{R}_{\alpha_3,\mu_3}^{\mu_4,\beta_3} & \mathcal{R}_{\alpha_4,\mu_4}^{\mu_5,\beta_4} & Y_{l-1,\mu_5}(\lambda_1) \end{array} \quad (15)$$

Note that the \mathcal{R} matrices in the k -th row (counted from the bottom) depend on λ_k . For an application of a similar technique see [18].

3 Reduction of a column.

We derive a relation between $\mathcal{O}_{k,l}^N$ and $\mathcal{O}_{k,l}^{N-1}$. To accomplish this we remove the last column of (15) by making use of (B.35)-(B.46). Concentrating on this last column we find

$$\begin{aligned}
& \begin{pmatrix} X_{l-L_s+1,\sigma} \\ \mathcal{R}_{\tau,\rho}^{\rho',\sigma} Y_{l-L_s,\rho'}(\lambda_{L_s}) \\ \vdots \\ \mathcal{R}_{\delta,\nu_3}^{\nu_3,\epsilon} Y_{l-k-1,\nu_3'}(\lambda_{k+1}) \\ \mathcal{R}_{\gamma,\nu_2}^{\nu_2,\delta} Y_{l-k,\nu_2'}(\lambda_k) \\ \mathcal{R}_{\beta_1,\nu_1}^{\nu_1,\gamma} Y_{l-k+1,\nu_1'}(\lambda_{k-1}) \\ \vdots \\ \mathcal{R}_{\alpha,\mu}^{\mu',\beta} Y_{l-1,\mu'}(\lambda_1) \end{pmatrix} = \sum_{k=1}^{L_s} f_k \begin{pmatrix} Y_{l+1-L_s,\rho}(\lambda_{L_s}) \\ \vdots \\ Y_{l-k,\nu_3}(\lambda_{k+1}) \\ X_{l+1-k,\nu_2}(\lambda_k) \\ Y_{l-k,\nu_1}(\lambda_{k-1}) \\ \vdots \\ Y_{l-2,\mu}(\lambda_1) \end{pmatrix} Y_{l-1,\alpha}(\eta) + f_{0X} \begin{pmatrix} Y_{l+1-L_s,\rho}(\lambda_{L_s}) \\ \vdots \\ Y_{l-k,\nu_3}(\lambda_{k+1}) \\ Y_{l-k+1,\nu_2}(\lambda_k) \\ Y_{l-k+2,\nu_1}(\lambda_{k-1}) \\ \vdots \\ Y_{l,\mu}(\lambda_1) \end{pmatrix} X_{l+1,\alpha}(\eta) \quad (16)
\end{aligned}$$

This describes the result of the action of the last column of \mathcal{R} -operators on the right column of Y -vectors in (15). In this process the last column of \mathcal{R} -operators disappears (in accordance with (B.35)-(B.42)) and the expression on the right hand side of (16) replaces the right column of Y -vectors in (15). The X -vector on top of the left hand side is converted to $Y_{l-1,\alpha}(\eta)$ and $X_{l+1,\alpha}(\eta)$ with free index α . We note that on the left hand side α is the only free quantum index, the other indices from β to σ are summed over whereas there are L_s free auxiliary indices from μ to ρ . This simplifies on account of the intertwining relations (B.35)-(B.42) to the direct products on the right side here written as columns in order to show clearly the origin of each factor Y . The coefficients $f_k, k = 0, \dots, L_s$ follow after repeated use of (B.35)-(B.42), but we will obtain them more transparently from (16). To extract f_m multiply (16) by

$$\tilde{X}_l(\lambda_1) \otimes \tilde{X}_{l-1}(\lambda_2) \otimes \dots \otimes \tilde{Y}_{l+1-m}(\lambda_m) \dots \otimes \tilde{X}_{l+1-L-1}(\lambda_L) \quad (17)$$

Note that all components are of type \tilde{X} except \tilde{Y}_{l-1+m} . Applying (B.17) and (B.18) we find

$$\begin{aligned}
& \begin{pmatrix} \tilde{X}_{l+1-L_s}^\rho(\lambda_{L_s}) \\ \vdots \\ \tilde{X}_{l-m}^{\nu_3}(\lambda_{m+1}) \\ \tilde{Y}_{l-m+1}^{\nu_2}(\lambda_m) \\ \tilde{X}_{l-m+2}^{\nu_1}(\lambda_{m-1}) \\ \vdots \\ \tilde{X}_l^\mu(\lambda_1) \end{pmatrix} \begin{pmatrix} X_{l-L_s+1,\sigma} \\ \mathcal{R}_{\tau,\rho}^{\rho',\sigma} Y_{l-L_s,\rho'}(\lambda_{L_s}) \\ \vdots \\ \mathcal{R}_{\delta,\nu_3}^{\nu_3,\epsilon} Y_{l-k-1,\nu_3'}(\lambda_{k+1}) \\ \mathcal{R}_{\gamma,\nu_2}^{\nu_2,\delta} Y_{l-k,\nu_2'}(\lambda_k) \\ \mathcal{R}_{\beta_1,\nu_1}^{\nu_1,\gamma} Y_{l-k+1,\nu_1'}(\lambda_{k-1}) \\ \vdots \\ \mathcal{R}_{\alpha,\mu}^{\mu',\beta} Y_{l-1,\mu'}(\lambda_1) \end{pmatrix} \\
& = f_m \prod_{r=0}^{m-2} (\tilde{X}_{l-r}(\lambda_{r+1}) Y_{l-r-2}(\lambda_{r+1})) \times \prod_{s=m}^{L_s} m(\lambda_s) Y_{l-1,\alpha}(\eta) \quad (18)
\end{aligned}$$

The left hand side follows after repeated application of (B.32)-(B.34). We find

$$\begin{aligned}
& \tilde{X}_{l+1-L_s}^\rho(\lambda_{L_s}) \quad \mathcal{R}_{\tau,\rho}^{\rho',\sigma} \quad Y_{l-L_s,\rho'}(\lambda_{L_s}) \\
& \vdots \\
& \tilde{X}_{l-m}^{\nu_3}(\lambda_{m+1}) \quad \mathcal{R}_{\delta,\nu_3}^{\nu_3',\epsilon} \quad Y_{l-k-1,\nu_3'}(\lambda_{k+1}) \\
& \tilde{Y}_{l-m+1}^{\nu_2}(\lambda_m) \quad \mathcal{R}_{\gamma,\nu_2}^{\nu_2',\delta} \quad Y_{l-k,\nu_2'}(\lambda_k) \\
& \tilde{X}_{l-m+2}^{\nu_1}(\lambda_{m-1}) \quad \mathcal{R}_{\beta_1,\nu_1}^{\nu_1',\gamma} \quad Y_{l-k+1,\nu_1'}(\lambda_{k-1}) \\
& \vdots \\
& \tilde{X}_l^\mu(\lambda_1) \quad \mathcal{R}_{\alpha,\mu}^{\mu',\beta} \quad Y_{l-1,\mu'}(\lambda_1)
\end{aligned} = \prod_{j=1}^{m-1} h(\lambda_j + \eta) \prod_{j=m+1}^{L_s} h(\lambda_j - \eta) \times \prod_{r=0}^{m-2} (\tilde{X}_{l-r}(\lambda_{r+1}) Y_{l-r-2}(\lambda_{r+1})) \times \prod_{s=1}^{L_s} m(\lambda_s) Y_{l-1,\alpha}(\eta) \quad (19)$$

Result:

$$f_k(l) = h(2\eta) \frac{g(\tau_{l-k+1} + \lambda_k - \eta)}{g(\tau_{l-k+1})} \prod_{j=1}^{k-1} h(\lambda_j + \eta) \prod_{j=k+1}^{L_s} h(\lambda_j - \eta) \quad (20)$$

$$f_{0X} = \prod_{j=1}^{L_s} h(\lambda_j - \eta) \quad (21)$$

To proceed further we insert (16) into (15).

$$\begin{aligned}
& B_{j+1,l-1}^N(\lambda_1) B_{j+2,l-2}^N(\lambda_2) \cdots B_{l+L_s,l-L_s}^N(\lambda_{L_s}) \Omega^N = \sum_{k=1}^{L_s} f_k \times \\
& B_{j+1,l-2}^{N-1}(\lambda_1) \cdots B_{j+k-1,l-k}^{N-1}(\lambda_{k-1}) A_{j+k,l-k+1}^{N-1}(\lambda_k) B_{j+k+1,l-k}^{N-1}(\lambda_{k+1}) \cdots B_{j+L_s,l+1-L_s}^{N-1}(\lambda_{L_s}) \Omega^{N-1} \otimes Y_{l-1}(\eta) \\
& + f_{0X} B_{j+1,l}^{N-1}(\lambda_1) B_{j+2,l-1}^{N-1}(\lambda_2) \cdots B_{j+L_s,l-L_s+1}^{N-1}(\lambda_{L_s}) \Omega^{N-1} \otimes X_{l+1}(\eta)
\end{aligned} \quad (22)$$

We commute the operator A using relations (B.49) and (B.51) until it is positioned directly in front of Ω .

$$\begin{aligned}
& B_{j+1,l-1}^N(\lambda_1) B_{j+2,l-2}^N(\lambda_2) \cdots B_{l+L_s,l-L_s}^N(\lambda_{L_s}) \Omega^N = \\
& \left\{ \sum_{k=1}^{L_s} (-1)^{L_s-k} f_k \prod_{m=k}^{L_s-1} \beta_{l-m}(\lambda_m, \lambda_{m+1}) \right\} \\
& B_{j+1,l-2}^{N-1}(\lambda_1) \cdots B_{j+L_s-1,l-L_s}^{N-1}(\lambda_{L_s-1}) A_{j+L_s,l-L_s+1}^{N-1}(\lambda_{L_s}) \Omega^{N-1} \otimes Y_{l-1}(\eta) + \\
& f_{0X} B_{j+1,l}^{N-1}(\lambda_1) B_{j+2,l-1}^{N-1}(\lambda_2) \cdots B_{j+L_s,l-L_s+1}^{N-1}(\lambda_{L_s}) \Omega^{N-1} \otimes X_{l+1}(\eta)
\end{aligned} \quad (23)$$

The observation that the expression in braces vanishes is fundamental to that what follows. This will allow us to obtain our result recursively. So we prove that

$$F = \sum_{k=1}^{L_s} (-1)^{L_s-k} f_k(l) \prod_{m=k}^{L_s-1} \beta_{l-m}(\lambda_m, \lambda_{m+1}) = 0. \quad (24)$$

We obtain from (B.51) that

$$F = g(\tau_{l-L_s}) \sum_{k=1}^{L_s} \frac{f_k(l)}{g(\tau_{l-k})} \quad (25)$$

and after insertion of (20)

$$F = h(2\eta) g(\tau_{l-L_s}) \sum_{k=1}^{L_s} \left\{ \prod_{j=1}^{k-1} h(\lambda_j + \eta) \right\} \frac{g(\tau_{l-k+1} + \lambda_k - \eta)}{g(\tau_{l-k+1}) g(\tau_{l-k})} \left\{ \prod_{j=k+1}^{L_s} h(\lambda_j - \eta) \right\} \quad (26)$$

$$F \prod_{r=0}^{L_s-1} h(\tau_{l-r}) = h(2\eta) \sum_{k=1}^{L_s} \prod_{j=1}^{k-1} h(\lambda_j + \eta) \prod_{j=k+1}^{L_s} h(\lambda_j - \eta) \prod_{r=0}^{k-2} h(\tau_{l-r}) \prod_{r=k+1}^L h(\tau_{l-r}) h(\tau_{l-k+1} + \lambda_k - \eta) \quad (27)$$

We now insert (6) to make use of the fact that the arguments λ_k form an exact string.

$$F \prod_{r=0}^{L_s-1} h(\tau_{l-r}) = h(2\eta) \sum_{k=1}^{L_s} h(\tau_l + \lambda_c - (4k-3)\eta) \times \quad (28)$$

$$\left\{ \prod_{j=1}^{k-1} h(\lambda_c - (2j-3)\eta) \right\} \left\{ \prod_{j=k+1}^{L_s} h(\lambda_c - (2j-1)\eta) \right\} \left\{ \prod_{r=0}^{k-2} h(\tau_l - 2r\eta) \right\} \left\{ \prod_{r=k+1}^L h(\tau_l - 2r\eta) \right\}$$

We write this as

$$F \prod_{r=0}^{L_s-1} h(\tau_{l-r}) = \sum_{k=1}^{L_s} p_k(\lambda_c) \quad (29)$$

A close inspection reveals that if for example λ_0 is a zero of $p_1(\lambda_c)$ then λ_0 is also a zero of $\sum_{k=2}^{L_s} p_k(\lambda_c)$. One finds that if λ_0 is a zero of p_1 all but two of the other p_k have the same zero λ_0 and that the remaining two terms cancel for $\lambda_c = \lambda_0$. Then $\sum_{k=2}^{L_s} \frac{p_k(\lambda_c)}{p_1(\lambda_c)}$ is doubly periodic and does not have poles. It follows that it is a constant. It is easily shown that this constant is -1 . This proves that

$$F = 0 \quad (30)$$

The corresponding column acting on a Y -type vector is

$$\begin{pmatrix} Y_{l-L_s-1,\sigma} \\ R_{\tau,\rho}^{\rho',\sigma} Y_{l-L_s,\rho'}(\lambda_{L_s}) \\ \vdots \\ \mathcal{R}_{\delta,\nu_3}^{\nu'_3,\epsilon} Y_{l-k-1,\nu'_3}(\lambda_{k+1}) \\ \mathcal{R}_{\gamma,\nu_2}^{\nu'_2,\delta} Y_{l-k,\nu'_2}(\lambda_k) \\ \mathcal{R}_{\beta_1,\nu_1}^{\nu'_1,\gamma} Y_{l-k+1,\nu'_1}(\lambda_{k-1}) \\ \vdots \\ \mathcal{R}_{\alpha,\mu}^{\mu',\beta} Y_{l-1,\mu'}(\lambda_1) \end{pmatrix} = f_{0Y} \begin{pmatrix} Y_{l-1-L_s,\rho}(\lambda_{L_s}) \\ \vdots \\ Y_{l-k-2,\nu_3}(\lambda_{k+1}) \\ Y_{l-k-1,\nu_2}(\lambda_k) \\ Y_{l-k,\nu_1}(\lambda_{k-1}) \\ \vdots \\ Y_{l-2,\mu}(\lambda_1) \end{pmatrix} Y_{l-1,\alpha}(\eta) \quad (31)$$

$$f_{0Y} = \prod_{j=1}^{L_s} h(\lambda_j + \eta) \quad (32)$$

The explanatory remarks following (16) apply also here. We note that

$$\prod_{j=1}^{L_s} h(\lambda_j + \eta) = \pm \prod_{j=1}^{L_s} h(\lambda_j - \eta) \quad (33)$$

where the minus sign holds if $2L_s\eta$ is an odd multiple of $2K$ and the plus sign if $2L_s\eta$ is a multiple of $4K$. Because of (24) (23) is considerably simplified:

$$\begin{aligned} & B_{j+1,l-1}^N(\lambda_1) B_{j+2,l-2}^N(\lambda_2) \cdots B_{j+L_s,l-L_s}^N(\lambda_{L_s}) \Omega^{N-1} \otimes X_{l+1-L_s}(\eta) = \\ & f_{0X} B_{j+1,l}^{N-1}(\lambda_1) B_{j+2,l-1}^{N-1}(\lambda_2) \cdots B_{j+L_s,l-L_s+1}^{N-1}(\lambda_{L_s}) \Omega^{N-1} \otimes X_{l+1,\alpha}(\eta) \end{aligned} \quad (34)$$

and from (31) follows directly

$$\begin{aligned} & B_{j+1,l-1}^N(\lambda_1) B_{j+2,l-2}^N(\lambda_2) \cdots B_{j+L_s,l-L_s}^N(\lambda_{L_s}) \Omega^{N-1} \otimes Y_{l-1-L_s}(\eta) = \\ & f_{0Y} B_{j+1,l-2}^{N-1}(\lambda_1) B_{j+2,l-3}^{N-1}(\lambda_2) \cdots B_{j+L_s,l-L_s-1}^{N-1}(\lambda_{L_s}) \Omega^{N-1} \otimes Y_{l-1,\alpha}(\eta) \end{aligned} \quad (35)$$

We have found that the string operator acting on a chain of length N is related to a string operator acting on a chain of length $N - 1$. This happens because the first column on the right hand side of (16) which would destroy this simple relationship does not contribute on account of (24).

4 The recursion process.

To continue we introduce a shorter notation. For a column headed by an X vector we write

$$\mathcal{C}_X(l) = \mathcal{R}(\lambda_1)_{\alpha\mu_1}^{\nu_1\beta_1} \mathcal{R}(\lambda_2)_{\beta_1\mu_2}^{\nu_2\beta_2} \cdots \mathcal{R}(\lambda_{L_s})_{\beta_{L_s-1}\mu_{L_s}}^{\nu_{L_s}\beta_{L_s}} X_{l\beta_{L_s}}(\eta) \quad (36)$$

and for a column headed by a Y vector

$$\mathcal{C}_Y(l) = \mathcal{R}(\lambda_1)_{\alpha\mu_2}^{\nu_1\beta_1} \mathcal{R}(\lambda_1)_{\beta_1\mu_2}^{\nu_2\beta_2} \cdots \mathcal{R}(\lambda_{L_s})_{\beta_{L_s-1}\mu_{L_s}}^{\nu_{L_s}\beta_{L_s}} Y_{l\beta_{L_s}}(\eta) \quad (37)$$

The direct products of Y and X as occurring in (16) will be abbreviated by

$$V_Y(l) = Y_l(\lambda_1) \otimes Y_{l-1}(\lambda_2) \otimes \cdots \otimes Y_{l-L_s+1}(\lambda_{L_s}) \quad (38)$$

$$V_{Y1X}(l, k) = Y_l(\lambda_1) \otimes Y_{l-1}(\lambda_2) \otimes \cdots \otimes Y_{l+2-k}(\lambda_{k-1}) \otimes X_{l+3-k}(\lambda_k) \otimes Y_{l+2-k}(\lambda_{k+1}) \otimes \cdots \otimes Y_{l+3-L_s}(\lambda_{L_s}) \quad (39)$$

where k marks the position of X . In this notation (16) and (31) read

$$\mathcal{C}_X(l - L_s + 1) V_Y(l - 1) = \sum_{k=1}^{L_s} f_k V_{Y1X}(l - 2, k) \otimes Y_{l-1}(\eta) + f_{0X} V_Y(l) \otimes X_{l+1}(\eta) \quad (40)$$

$$\mathcal{C}_Y(l - L_s - 1) V_Y(l - 1) = f_{0Y} V_Y(l - 2) \otimes Y_{l-1}(\eta) \quad (41)$$

We use the results obtained in the last section to study the action of B-strings on the set of vectors described in (12). These states have the local structure

$$\begin{aligned} & \cdots \otimes X_{l+1}(\eta) \otimes X_l(\eta) \otimes \cdots \\ & \cdots \otimes Y_{l-1}(\eta) \otimes X_l(\eta) \otimes \cdots \\ & \cdots \otimes X_{l+1}(\eta) \otimes Y_l(\eta) \otimes \cdots \\ & \cdots \otimes Y_{l-1}(\eta) \otimes Y_l(\eta) \otimes \cdots \end{aligned} \quad (42)$$

i.e. the indices of neighboring vectors differ by ± 1 depending on their order as shown in (42). This rule for the indices of consecutive X and Y guarantees that relation (B.36) is in all cases applicable. In a system of length N these vectors span a linear space of dimension 2^N . They are obviously closely related to the family of vectors introduced by Baxter in [4] (1.3). Then

$$\begin{aligned} & \left(\prod_{k=1}^{L_s} m(\lambda_k) \right) B_{l+1, l-1}^N(\lambda_1) B_{l+2, l-2}^N(\lambda_2) \cdots B_{l+L_s, l-L_s}^N(\lambda_{L_s}) \Omega^N = \\ & V_{\hat{Y}}(l+1) \mathcal{C}_{Z_1}(l_1) \cdots \mathcal{C}_{Z_{N-1}}(l_{N-1}) \mathcal{C}_X(l - L_s + 1) V_Y(l - 1) \end{aligned} \quad (43)$$

if $\Omega^N = \Omega^{N-1} \otimes X_{l+1-L_s}(\eta)$ and

$$\left(\prod_{k=1}^{L_s} m(\lambda_k) \right) B_{l+1,l-1}^N(\lambda_1) B_{l+2,l-2}^N(\lambda_2) \cdots B_{l+L_s,l-L_s}^N(\lambda_{L_s}) \Omega^N =$$

$$V_{\tilde{Y}}(l+1) \mathcal{C}_{Z_1}(l_1) \cdots \mathcal{C}_{Z_{N-1}}(l_{N-1}) \mathcal{C}_Y(l-L_s-1) V_Y(l-1) \quad (44)$$

if $\Omega^N = \Omega^{N-1} \otimes Y_{l-1-L_s}(\eta)$

We have shown (see (23) and (24)) that (40) and (41) if inserted into B-strings effectively read

$$\mathcal{C}_X(l-L_s+1) V_Y(l-1) = f_{0X} V_Y(l) \otimes X_{l+1}(\eta) \quad (45)$$

$$\mathcal{C}_Y(l-L_s-1) V_Y(l-1) = f_{0Y} V_Y(l-2) \otimes Y_{l-1}(\eta) \quad (46)$$

where $f_{0Y} = \pm f_{0X}$

4.1 The leftmost column.

The last step in the recursive determination is the treatment of the first column.

Case a:

If

$$\Omega^N = X_{l+n_x-n_y-L_s} \otimes \cdots \otimes X_{l+1-L_s} \quad \text{or} \quad \Omega^N = X_{l+n_x-n_y-L_s} \otimes \cdots \otimes Y_{l-1-L_s} \quad (47)$$

the first column is

$$\mathcal{C}_X(l+n_x-n_y-L_s) V_Y(l+n_x-n_y-2) \quad (48)$$

Case b:

and if

$$\Omega^N = Y_{l+n_x-n_y-L_s} \otimes \cdots \otimes X_{l+1-L_s} \quad \text{or} \quad \Omega^N = Y_{l+n_x-n_y-L_s} \otimes \cdots \otimes Y_{l-1-L_s} \quad (49)$$

the first column is

$$\mathcal{C}_Y(l+n_x-n_y-L_s) V_Y(l+n_x-n_y) \quad (50)$$

In (47) and (49) the gap between the leading and the trailing X or Y is filled with vectors having indices according to the rule illustrated by (42).

4.1.1 Case a.

From (40) follows

$$V_{\tilde{Y}}(l+1) \mathcal{C}_X(l+n_x-n_y-L_s) V_Y(l+n_x-n_y-2) =$$

$$\sum_{k=1}^{L_s} f_k(l+M-1) (V_{\tilde{Y}}(l+1) V_{Y_1 X}(l+M-3, k)) \otimes Y_{l+M-2}(\eta)$$

$$+ f_{0X} (V_{\tilde{Y}}(l+1) V_Y(l+M-1)) \otimes X_{l+M}(\eta) \quad (51)$$

where $M = n_x - n_y$

$$(V_{\tilde{Y}}(l+1)V_{Y1X}(l+M-3, k)) = \left\{ \prod_{m=1}^{k-1} \tilde{Y}_{l+m}(\lambda_m) Y_{l+M-2-m}(\lambda_m) \right\} \tilde{Y}_{l+k}(\lambda_k) X_{l+M-k}(\lambda_k) \left\{ \prod_{m=k+1}^{L_s} \tilde{Y}_{l+m}(\lambda_m) Y_{l+M-m}(\lambda_m) \right\} \quad (52)$$

This expression is most conveniently handled once we recognize that it is

$$(V_{\tilde{Y}}(l+1)V_{Y1X}(l+M-3, k)) = \prod_{k=1}^{L_s} m(\lambda_k) \quad (53)$$

$$\left\{ \prod_{m=1}^{k-1} B_{l+m, l+M-2-m}^{(0)}(\lambda_m) \right\} A_{l+k, l+M-k}^{(0)}(\lambda_k) \left\{ \prod_{m=k+1}^{L_s} B_{l+m, l+M-m}^{(0)}(\lambda_m) \right\} \quad (54)$$

for a chain of length zero and that the permutation relations for A and B still hold in this case (see (B.53) in Appendix B).

$$(V_{\tilde{Y}}(l+1)V_{Y1X}(l+M-3, k)) = \prod_{k=1}^{L_s} m(\lambda_k) \quad (55)$$

$$(-1)^{L_s-k} \prod_{k+1}^{L_s} \beta_{l+M-m}(\lambda_{m-1}, \lambda_m) \prod_{m=1}^{L_s-1} B_{l+m, l+M-2-m}^{(0)}(\lambda_m) A_{l+L_s, l+M-L_s}^{(0)}(\lambda_{L_s}) \quad (56)$$

Insert $\beta_{l+M-m}(\lambda_{m-1}, \lambda_m)$ using (B.51)

$$(V_{\tilde{Y}}(l+1)V_{Y1X}(l+M-3, k)) = \prod_{k=1}^{L_s} m(\lambda_k) \quad (57)$$

$$\frac{g(\tau_{l+M-L_s-1})}{g(\tau_{l+M-k-1})} \prod_{m=1}^{L_s-1} B_{l+m, l+M-2-m}^{(0)}(\lambda_m) A_{l+L_s, l+M-L_s}^{(0)}(\lambda_{L_s}) \quad (58)$$

It follows for the first term on the right hand side of (52)

$$\begin{aligned} \sum_{k=1}^{L_s} f_k(l+M-1) (V_{\tilde{Y}}(l+1)V_{Y1X}(l+M-3, k)) \otimes Y_{l+M-2}(\eta) &= \prod_{k=1}^{L_s} m(\lambda_k) \\ \sum_{k=1}^{L_s} f_k(l+M-1) \frac{g(\tau_{l+M-L_s-1})}{g(\tau_{l+M-k-1})} \prod_{m=1}^{L_s-1} B_{l+m, l+M-2-m}^{(0)}(\lambda_m) A_{l+L_s, l+M-L_s}^{(0)}(\lambda_{L_s}) &\otimes Y_{l+M-2}(\eta) \end{aligned} \quad (59)$$

We have already shown that (see (25))

$$\sum_{k=1}^{L_s} f_k(l+M-1) \frac{g(\tau_{l+M-L_s-1})}{g(\tau_{l+M-k-1})} = 0 \quad (60)$$

Therefore (52) is reduced to

$$V_{\tilde{Y}}(l+1)\mathcal{C}_X(l+n_x-n_y-L_s)V_Y(l+n_x-n_y-2) = f_{0X}(V_{\tilde{Y}}(l+1)V_Y(l+M-1)) \otimes X_{l+M}(\eta) \quad (61)$$

We claim that for even N

$$(V_{\tilde{Y}}(l+1)V_Y(l+M-1)) = \prod_{k=1}^{L_s} \tilde{Y}_{l+k}^{\mu}(\lambda_k) Y_{l+M-k,\mu}(\lambda_k) = 0 \quad (62)$$

The last expression vanishes if for some $1 \leq k \leq L_s$

$$2k\eta = (M-k)2\eta + 4rK \quad (63)$$

or

$$k = M/2 + r_1 L_s \quad (64)$$

For even N also M is even and by appropriately adjusting r_1 an integer k is found which satisfies (63).

4.1.2 Case b.

From (41) follows

$$V_{\tilde{Y}}(l+1)\mathcal{C}_Y(l+M-L)V_Y(l+M) = f_{0Y}(V_{\tilde{Y}}(l+1)V_Y(l+M-1)) \otimes Y_{l+M}(\eta) \quad (65)$$

In the same manner as above we can show that this is zero if M is even. This completes the proof of equation (8) for even N .

$$B_{l+1,l-1}(\lambda_1) \cdots B_{l+L_s,l-L_s}(\lambda_{L_s}) = 0 \quad (66)$$

5 Chains of odd length.

We remark that in the algebraic Bethe Ansatz eigenvectors are obtained for even N . The eight vertex model for chains of odd length N is nevertheless interesting as demonstrated in [17], [19], [20]. We therefore present also our result for odd N which surprisingly shows that the string operator does

not vanish but is proportional to a simple symmetry operator. In this case we have to collect the contributions of each column in the stepwise reduction of a system of size $= N$ to a system of size $= 1$ and to multiply finally by the result for the remaining system of size $= 1$ obtained in the preceding section. We observe that the explicit form of X and Y given in (B.13) and (B.14) shows that for integer m

$$X_{l-L_s}(\eta) = -\sigma_3 X_l(\eta) \quad Y_{l-L_s}(\eta) = \sigma_3 Y_l(\eta) \quad \text{if } 2L_s\eta = (2m+1)2K \quad (67)$$

$$X_{l-L_s}(\eta) = X_l(\eta) \quad Y_{l-L_s}(\eta) = Y_l(\eta) \quad \text{if } 2L_s\eta = m4K \quad (68)$$

5.1 The case $2L_s\eta = (2m+1)2K$.

We conclude from (34), (35), (52), (65) and (67) that the action of the string operator B_s on a state $\Omega(n_X, n_Y)$ characterized in (12) and (42) where n_X and n_Y denote the number of X - and Y -vectors is given by

$$B_{l+1, l-1}(\lambda_1) \cdots B_{l+L_s, l-L_s}(\lambda_{L_s}) \Omega(n_X, n_Y) = - \left(\prod_{k=1}^{L_s} m(\lambda_k)^{-1} \right) C(\lambda_c, M) f_{0X}^N \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 \Omega(n_X, n_Y) \quad (69)$$

where

$$C(\lambda_c, M) = (V_{\tilde{Y}}(l+1) V_Y(l+M-1)) = \prod_{k=1}^{L_s} \tilde{Y}_{l+k}^\mu(\lambda_k) Y_{l+M-k, \mu}(\lambda_k) \quad (70)$$

and where (67) and $f_{0Y} = -f_{0X}$ as well as $n_X + n_Y = N$ and that N is odd have been taken into account.

5.2 The case $2L_s\eta = m4K$

We use (68) and $f_{0Y} = +f_{0X}$:

$$B_{l+1, l-1}(\lambda_1) \cdots B_{l+L_s, l-L_s}(\lambda_{L_s}) \Omega(n_X, n_Y) = \left(\prod_{k=1}^{L_s} m(\lambda_k)^{-1} \right) C(\lambda_c, M) f_{0X}^N \Omega(n_X, n_Y) \quad (71)$$

We claim that $C(\lambda_c, M)$ does not depend on $M = n_X - n_Y$.

Proof: From (70) and appendix B follows

$$C(\lambda, M) = \prod_{k=1}^{L_s} \frac{1}{g(\tau_{l+k}) g(\tau_{l+M-k})} (\Theta(u) \mathbf{H}(v) - \mathbf{H}(u) \Theta(v)) \quad (72)$$

with

$$u = t + 2(l + k)\eta + \lambda_k \quad v = t + 2(l + M - k)\eta + \lambda_k \quad \lambda_k = \lambda_c - 2(k - 1)\eta \quad (73)$$

We rewrite this by using the relations of appendix A

$$C(\lambda, M) = \frac{2}{\mathbf{H}(K)\Theta(K)} \prod_{k=1}^{L_s} \frac{1}{g(\tau_{l+k})g(\tau_{l+M-k})} \mathbf{H}((M/2 - k)2\eta)\Theta((M/2 - k)2\eta) \times \\ \mathbf{H}(t + \lambda_c + (l + 1 + M/2 - k)2\eta + K)\Theta(t + \lambda_c + (l + 1 + M/2 - k)2\eta + K) \quad (74)$$

We compare $C(\lambda, M)$ with $C(\lambda, M + 2)$. If $\eta = 2m_1 K/L$

$$\prod_{k=1}^{L_s} g(\tau_{l+M+2-k}) = \prod_{k=-1}^{L_s-2} g(\tau_{l+M-k}) = \prod_{k=1}^{L_s} g(\tau_{l+M-k}) \quad (75)$$

as an even number of shifts by $2K$ occur. Similarly a shift of M by 2 is compensated by a shift of k by 1 in

$$\prod_{k=1}^{L_s} \mathbf{H}((M/2 - k)2\eta)\Theta((M/2 - k)2\eta)\mathbf{H}(t + \lambda_c + (l + 1 + M/2 - k)2\eta + K)\Theta(t + \lambda_c + (l + 1 + M/2 - k)2\eta + K) \quad (76)$$

It follows

$$C(\lambda_c, M + 2) = C(\lambda_c, M) \quad (77)$$

and as M is odd the simplest choice is

$$C(\lambda_c, M) = C(\lambda_c, -1) = \prod_{k=1}^{L_s} \tilde{Y}_{l+k}^\mu(\lambda_k) Y_{l-1-k, \mu}(\lambda_k) \quad (78)$$

Because of the presence of $\mathbf{H}((M/2 - k)2\eta)$ in (76) $C(\lambda, M)$ vanishes if $2L_s\eta = m4K$ for integer m . However it does *not vanish* if $2L_s\eta = (2m + 1)2K$. This means that for odd N $B_{l+1, l-1}(\lambda_1) \cdots B_{l+L_s, l-L_s}(\lambda_{L_s})$ is proportional to the operator $S = \sigma_3 \otimes \sigma_3 \cdots \otimes \sigma_3$.

6 Summary of Results

We have studied the properties of the operator

$$B_s(\lambda_c) = B_{l+1, l-1}(\lambda_1) \cdots B_{l+L_s, l-L_s}(\lambda_{L_s}) \quad (79)$$

at roots of unity

$$\eta = 2m_1 K/L \quad (80)$$

1. For even N : $B_s = 0$.
2. For odd N and $2L_s\eta \equiv 0 \pmod{4K}$: $B_s = 0$
3. For odd N and $2L_s\eta \equiv 2K \pmod{4K}$:

$$B_{l+1,l-1}(\lambda_1) \cdots B_{l+L_s,l-L_s}(\lambda_{L_s}) \Omega(n_X, n_Y) = f(\lambda_c) \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3 \Omega(n_X, n_Y) \quad (81)$$

$$f(\lambda_c) = - \left(\prod_{k=1}^{L_s} m(\lambda_k)^{-1} \right) f_{0X}^N \prod_{k=1}^{L_s} \tilde{Y}_{l+k}^\mu(\lambda_k) Y_{l-1-k,\mu}(\lambda_k) \quad (82)$$

To clarify what happens for odd N we give some examples.

If $\eta = K/3$ then because of (80) $L = 6$ and $L_s = 3$, $2L_s\eta = 2K$ and (81) applies.

If $\eta = 2K/3$ then $L = 3$ and $L_s = 3$, $2L_s\eta = 4K$ and $B_s = 0$

If $\eta = K/2$ then $L = 4$ and $L_s = 2$, $2L_s\eta = 2K$ and (81) applies.

7 Conclusions

A common feature of the six vertex model and the eight vertex model for crossing parameters at roots of unity is the occurrence of degeneracies in the eigenvalues of the transfer matrix T and the Hamiltonian H . This is by no means an exceptional case, because analytic solutions of the eight-vertex model which include the eigenvectors of the transfer matrix T exist only for root of unity η . For the six vertex model the underlying symmetry algebra is fairly well understood. It has been shown in [12] that the sl_2 loop algebra symmetry is responsible for the degeneracies of eigenvalues and its Chevalley generators have been explicitly constructed. The picture was completed by the construction of the generating function of the operators in the mode basis (the current) in [13] which allowed the determination of the evaluation parameters. This current plays two different roles: it generates the sl_2 loop algebra operators in the mode basis and it is the operator which creates exact Bethe strings in the algebraic Bethe ansatz. It turns out that the set of B operators is itself not complete in the sense that it is not capable of creating the complete eigenstate of T . To achieve this the current operators have to be incorporated.

We now turn to the eight vertex model. The result of this paper given in the summary completes the construction of the operator which generates complete Bethe strings and is in this respect a generalization of the six vertex current operator introduced in [13]. Expression (5) is certainly a symmetry operator as it maps degenerate subspaces onto itself. But the symmetry operators which generalize the Chevalley operators found and studied in [12] are still elusive.

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Appendix A. Theta functions

The definition of Jacobi Theta functions of nome q (see (21.62) [21]) is

$$\mathbf{H}(v) = 2 \sum_{n=1}^{\infty} (-1)^{n-1} q^{(n-1/2)^2} \sin[(2n-1)\pi v/(2K)] \quad (\text{A.1})$$

$$\Theta(v) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos(nv\pi/K) \quad (\text{A.2})$$

where K and K' are the standard elliptic integrals of the first kind and

$$q = \exp -\pi K'/K. \quad (\text{A.3})$$

These theta functions satisfy the quasi periodicity relations

$$\mathbf{H}(v+2K) = -\mathbf{H}(v) \quad \mathbf{H}(v+2iK') = -q^{-1} \exp(-\pi i v/K) \mathbf{H}(v) \quad (\text{A.4})$$

$$\Theta(v+2K) = \Theta(v) \quad \Theta(v+2iK') = -q^{-1} \exp(-\pi i v/K) \Theta(v). \quad (\text{A.5})$$

$\Theta(v)$ and $\mathbf{H}(v)$ are related by

$$\Theta(v+iK') = i q^{-1/4} \exp -\frac{\pi i v}{2K} \mathbf{H}(v) \quad \mathbf{H}(v+iK') = i q^{-1/4} \exp -\frac{\pi i v}{2K} \Theta(v). \quad (\text{A.6})$$

They satisfy the addition theorem

$$\mathbf{H}(u)\Theta(v) - \Theta(u)\mathbf{H}(v) = 2\mathbf{H}((u-v)/2)\Theta((u-v)/2)\mathbf{H}((u+v)/2+K)\Theta((u+v)/2+K)/(\mathbf{H}(K)\Theta(K)) \quad (\text{A.7})$$

Appendix B. The algebraic Bethe ansatz

We only list those definitions and identities which we make use of. We follow the formalism of [9]. The monodromy matrix is defined as

$$\mathcal{T} = \mathcal{L}_N \cdots \mathcal{L}_1 = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad (\text{B.1})$$

where \mathcal{L}_n is a 2×2 matrix in auxiliary space with entries which are 2×2 matrices in spin space acting on the n th spin in the spin chain and A, B, C, D are $2^N \times 2^N$ matrices in spin space.

$$\mathcal{L}_n = \begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix} \quad (\text{B.2})$$

$$\alpha_n = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad \beta_n = \begin{pmatrix} 0 & d \\ c & 0 \end{pmatrix}, \quad \gamma_n = \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix}, \quad \delta_n = \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix} \quad (\text{B.3})$$

a, b, c and d are defined in (2). The R matrix is

$$\mathcal{R}_{\alpha, \kappa}^{\rho, \beta}(\lambda, \mu) = \frac{1}{2}(a_R + b_R)E \otimes E + \frac{1}{2}(a_R - b_R)\sigma^3 \otimes \sigma^3 + c_R(\sigma^- \otimes \sigma^+ + \sigma^+ \otimes \sigma^-) + d_R(\sigma^+ \otimes \sigma^+ + \sigma^- \otimes \sigma^-) \quad (\text{B.4})$$

where on the right hand side the first factor of the direct products carries the indices α, ρ and the second factor κ, β .

$$a_R = \Theta(2\eta)\Theta(\lambda - \mu)\text{H}(\lambda - \mu + 2\eta) \quad (\text{B.5})$$

$$b_R = \text{H}(2\eta)\Theta(\lambda - \mu)\Theta(\lambda - \mu + 2\eta) \quad (\text{B.6})$$

$$c_R = \Theta(2\eta)\text{H}(\lambda - \mu)\Theta(\lambda - \mu + 2\eta) \quad (\text{B.7})$$

$$d_R = \text{H}(2\eta)\text{H}(\lambda - \mu)\text{H}(\lambda - \mu + 2\eta) \quad (\text{B.8})$$

$$a = a_R(\lambda, \eta) \quad b = c_R(\lambda, \eta) \quad c = b_R(\lambda, \eta) \quad d = d_R(\lambda, \eta) \quad (\text{B.9})$$

and the local transition matrix (B.2),(B.3) is

$$\mathcal{L}_{\alpha, \kappa}^{\beta, \rho}(\lambda) = \mathcal{R}_{\alpha, \kappa}^{\rho, \beta}(\lambda, \eta) \quad (\text{B.10})$$

In order to construct a generating vector suitable for the eight-vertex model a gauge transformed monodromy matrix is defined as

$$\mathcal{T}_{k,l} = M_k^{-1}(\lambda)\mathcal{T}(\lambda)M_l(\lambda) = \begin{pmatrix} A_{k,l} & B_{k,l} \\ C_{k,l} & D_{k,l} \end{pmatrix}. \quad (\text{B.11})$$

where the matrices M_k introduced by Baxter [3] are

$$M_k = \begin{pmatrix} x_k^1 & y_k^1 \\ x_k^2 & y_k^2 \end{pmatrix} \quad M_k^{-1} = \frac{1}{m(\lambda)} \begin{pmatrix} y_k^2 & -y_k^1 \\ -x_k^2 & x_k^1 \end{pmatrix} \quad (\text{B.12})$$

The columns of M and rows of M^{-1} , called intertwining vectors ([18]) are of central importance:

$$X_k(\lambda) = \begin{pmatrix} x_k^1(\lambda) \\ x_k^2(\lambda) \end{pmatrix} = \begin{pmatrix} \text{H}(s + 2k\eta - \lambda) \\ \Theta(s + 2k\eta - \lambda) \end{pmatrix} \quad (\text{B.13})$$

$$Y_k(\lambda) = \begin{pmatrix} y_k^1(\lambda) \\ y_k^2(\lambda) \end{pmatrix} = \frac{1}{g(\tau_k)} \begin{pmatrix} \mathbf{H}(t + 2k\eta + \lambda) \\ \Theta(t + 2k\eta + \lambda) \end{pmatrix} \quad (\text{B.14})$$

$$g(u) = \mathbf{H}(u)\Theta(u), \quad \tau_l = (s + t)/2 + 2l\eta - K \quad (\text{B.15})$$

$$\tilde{Y}_k(\lambda) = (y_k^2(\lambda), -y_k^1(\lambda)) \quad \tilde{X}_k(\lambda) = (-x_k^2(\lambda), x_k^1(\lambda)) \quad (\text{B.16})$$

$$(\tilde{X}_k^\mu(\lambda)X_{k,\mu}(\lambda)) = 0 \quad (\tilde{Y}_k^\mu(\lambda)Y_{k,\mu}(\lambda)) = 0 \quad (\text{B.17})$$

$$(\tilde{X}_k^\mu(\lambda)Y_{k,\mu}(\lambda)) = m(\lambda) \quad (\tilde{Y}_k^\mu(\lambda)X_{k,\mu}(\lambda)) = m(\lambda) \quad (\text{B.18})$$

$$m(\lambda) = \frac{2g(\lambda + (t - s)/2)}{g(K)} \quad (\text{B.19})$$

The local vectors X_k defined in (B.13) generalize the local vacuum of the six vertex model

$$e^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (\text{B.20})$$

to the eight vertex model and build up the generating vectors (4.18) in [9]

$$\Omega_N^l = \omega_1^l \otimes \cdots \otimes \omega_N^l \quad (\text{B.21})$$

$$\omega_k^l = X_{k+l}(\eta) \quad (\text{B.22})$$

The elements of $\mathcal{T}_{k,l}$ are

$$A_{k,l}(\lambda) = \frac{1}{m(\lambda)} \tilde{Y}_k(\lambda) \mathcal{T}(\lambda) X_l(\lambda) \quad (\text{B.23})$$

$$B_{k,l}(\lambda) = \frac{1}{m(\lambda)} \tilde{Y}_k(\lambda) \mathcal{T}(\lambda) Y_l(\lambda) \quad (\text{B.24})$$

$$C_{k,l}(\lambda) = \frac{1}{m(\lambda)} \tilde{X}_k(\lambda) \mathcal{T}(\lambda) X_l(\lambda) \quad (\text{B.25})$$

$$D_{k,l}(\lambda) = \frac{1}{m(\lambda)} \tilde{X}_k(\lambda) \mathcal{T}(\lambda) Y_l(\lambda) \quad (\text{B.26})$$

The gauge transformed local transition matrix is

$$\mathcal{L}^l = M_l^{-1}(\lambda) \mathcal{L}(\lambda) M_{l-1}(\lambda) = \begin{pmatrix} \alpha^l & \beta^l \\ \gamma^l & \delta^l \end{pmatrix}. \quad (\text{B.27})$$

$$\alpha^l(\lambda) = \frac{1}{m(\lambda)} \tilde{Y}_l^\mu(\lambda) \mathcal{L}_{\alpha,\mu}^{\beta,\nu}(\lambda) X_{l-1,\nu}(\lambda) = \frac{1}{m(\lambda)} \tilde{Y}_l^\mu(\lambda) \mathcal{R}(\lambda, \eta)_{\alpha,\mu}^{\nu,\beta} X_{l-1,\nu}(\lambda) \quad (\text{B.28})$$

$$\beta^l(\lambda) = \frac{1}{m(\lambda)} \tilde{Y}_l^\mu(\lambda) \mathcal{L}_{\alpha,\mu}^{\beta,\nu}(\lambda) Y_{l-1,\nu}(\lambda) = \frac{1}{m(\lambda)} \tilde{Y}_l^\mu(\lambda) \mathcal{R}(\lambda, \eta)_{\alpha,\mu}^{\nu,\beta} Y_{l-1,\nu}(\lambda) \quad (\text{B.29})$$

$$\gamma^l(\lambda) = \frac{1}{m(\lambda)} \tilde{X}_l^\mu(\lambda) \mathcal{L}_{\alpha,\mu}^{\beta,\nu}(\lambda) X_{l-1,\nu}(\lambda) = \frac{1}{m(\lambda)} \tilde{X}_l^\mu(\lambda) \mathcal{R}(\lambda, \eta)_{\alpha,\mu}^{\nu,\beta} X_{l-1,\nu}(\lambda) \quad (\text{B.30})$$

$$\delta^l(\lambda) = \frac{1}{m(\lambda)} \tilde{X}_l^\mu(\lambda) \mathcal{L}_{\alpha,\mu}^{\beta,\nu}(\lambda) Y_{l-1,\nu}(\lambda) = \frac{1}{m(\lambda)} \tilde{X}_l^\mu(\lambda) \mathcal{R}(\lambda, \eta)_{\alpha,\mu}^{\nu,\beta} Y_{l-1,\nu}(\lambda) \quad (\text{B.31})$$

Action of $\alpha^l, \beta^l, \gamma^l, \delta^l$

To compute the left hand side of equ. (18) we need the relations

$$\delta^l(\lambda) X_l(\eta) = h(\lambda - \eta) X_{l+1}(\eta) \quad (\text{B.32})$$

$$\beta^l(\lambda) X_l(\eta) = h(2\eta) \frac{g(\tau_l + \lambda - \eta)}{g(\tau_l)} Y_{l-1}(\eta) \quad (\text{B.33})$$

$$\delta^l(\lambda) Y_{l-2}(\eta) = h(\lambda + \eta) \frac{(\tilde{X}_l^\mu(\lambda) Y_{l-2,\mu}(\lambda))}{m(\lambda)} Y_{l-1}(\eta) \quad (\text{B.34})$$

They follow from (B.35)-(B.46). Equ. (B.32) is identical with equ. (4.16) of [9].

Intertwining vectors

We make extensive use of the following powerful relations describing the action of the R-matrix on intertwining vectors [3], [9].

$$\mathcal{R}(\lambda, \mu)(X_l(\lambda) \otimes X_{l+1}(\mu)) = h(\lambda - \mu + 2\eta) X_l(\mu) \otimes X_{l+1}(\lambda) \quad (\text{B.35})$$

$$\mathcal{R}(\lambda, \mu)(Y_{l+1}(\lambda) \otimes Y_l(\mu)) = h(\lambda - \mu + 2\eta) Y_{l+1}(\mu) \otimes Y_l(\lambda) \quad (\text{B.36})$$

$$\mathcal{R}(\lambda, \mu)(Y_k(\lambda) \otimes X_l(\mu)) = f_{1,k,l}^{YX}(\lambda - \mu) Y_k(\mu) \otimes X_l(\lambda) + f_{2,k,l}^{YX}(\lambda - \mu) X_{l+1}(\mu) \otimes Y_{k+1}(\lambda) \quad (\text{B.37})$$

$$\mathcal{R}(\lambda, \mu)(X_k(\lambda) \otimes Y_l(\mu)) = f_{1,k,l}^{XY}(\mu - \lambda) X_k(\mu) \otimes Y_l(\lambda) + f_{2,k,l}^{XY}(\lambda - \mu) Y_{l-1}(\mu) \otimes X_{k-1}(\lambda) \quad (\text{B.38})$$

$$(\tilde{Y}_l(\mu) \otimes \tilde{Y}_{l+1}(\lambda)) \mathcal{R}(\lambda, \mu) = h(\lambda - \mu + 2\eta) \tilde{Y}_l(\lambda) \otimes \tilde{Y}_{l+1}(\mu) \quad (\text{B.39})$$

$$(\tilde{X}_{l+1}(\mu) \otimes \tilde{X}_l(\lambda)) \mathcal{R}(\lambda, \mu) = h(\lambda - \mu + 2\eta) \tilde{X}_{l+1}(\lambda) \otimes \tilde{X}_l(\mu) \quad (\text{B.40})$$

$$(\tilde{X}_k(\mu) \otimes \tilde{Y}_l(\lambda)) \mathcal{R}(\lambda, \mu) = f_{1,l,k}^{YX}(\lambda - \mu) \tilde{X}_k(\lambda) \otimes \tilde{Y}_l(\mu) + f_{2,l,k}^{YX}(\lambda - \mu) \tilde{Y}_{l+1}(\lambda) \otimes \tilde{X}_{k+1}(\mu) \quad (\text{B.41})$$

$$(\tilde{Y}_k(\mu) \otimes \tilde{X}_l(\lambda)) \mathcal{R}(\lambda, \mu) = f_{1,l,k}^{XY}(\mu - \lambda) \tilde{Y}_k(\lambda) \otimes \tilde{X}_l(\mu) + f_{2,l,k}^{XY}(\lambda - \mu) \tilde{X}_{l-1}(\lambda) \otimes \tilde{Y}_{k-1}(\mu) \quad (\text{B.42})$$

with

$$f_{1,k,l}^{YX}(\lambda - \mu) = \frac{h(2\eta)g(\tau_{(k+l+1)/2} + \lambda - \mu)}{g(\tau_{(k+l+1)/2})} \quad (\text{B.43})$$

$$f_{2,k,l}^{YX}(\lambda - \mu) = \frac{h(\lambda - \mu)g(\tau_{(k+l-1)/2})g(\tau_{k+1})}{g(\tau_{(k+l+1)/2})g(\tau_k)} \quad (\text{B.44})$$

$$f_{1,k,l}^{XY}(\mu - \lambda) = \frac{h(2\eta)g(\tau_{(k+l-1)/2} + \mu - \lambda)}{g(\tau_{(k+l-1)/2})} \quad (\text{B.45})$$

$$f_{2,k,l}^{XY}(\lambda - \mu) = \frac{h(\lambda - \mu)g(\tau_{(k+l+1)/2})g(\tau_{l-1})}{g(\tau_{(k+l-1)/2})g(\tau_l)} \quad (\text{B.46})$$

Permutation relations for $A_{k,l}, \dots, D_{k,l}$

\mathcal{T} and \mathcal{R} defined in (B.1) and (B.4) satisfy the RTT equation

$$\mathcal{R}(\lambda, \mu)(\mathcal{T}(\lambda) \otimes \mathcal{T}(\mu)) = (\mathcal{T}(\mu) \otimes \mathcal{T}(\lambda))\mathcal{R}(\lambda, \mu) \quad (\text{B.47})$$

which combined with (B.35)-(B.42) leads to the following fundamental permutation relations:

$$B_{k,l+1}(\lambda)B_{k+1,l}(\mu) = B_{k,l+1}(\mu)B_{k+1,l}(\lambda) \quad (\text{B.48})$$

$$A_{k,l}(\lambda)B_{k+1,l-1}(\mu) = \alpha(\lambda, \mu)B_{k,l-2}(\mu)A_{k+1,l-1}(\lambda) - \beta_{l-1}(\lambda, \mu)B_{k,l-2}(\lambda)A_{k+1,l-1}(\mu) \quad (\text{B.49})$$

$$D_{k,l}(\lambda)B_{k+1,l-1}(\mu) = \alpha(\mu, \lambda)B_{k+2,l}(\mu)D_{k+1,l-1}(\lambda) + \beta_{k+1}(\lambda, \mu)B_{k+2,l}(\lambda)D_{k+1,l-1}(\mu) \quad (\text{B.50})$$

where

$$\alpha(\lambda, \mu) = \frac{h(\lambda - \mu - 2\eta)}{h(\lambda - \mu)}, \quad \text{and} \quad \beta_k(\lambda, \mu) = \frac{h(2\eta)h(\tau_k + \mu - \lambda)}{h(\tau_k)h(\mu - \lambda)} \quad (\text{B.51})$$

and where

$$h(u) = \Theta(0)\Theta(u)\mathbf{H}(u) \quad (\text{B.52})$$

As follows from (B.35)-(B.46) these relations continue to be formally valid for a chain of length zero. We will need equ. (B.49) in this limit:

$$\begin{aligned} & (\tilde{Y}_k(\lambda)X_l(\lambda))(\tilde{Y}_{k+1}(\mu)Y_{l-1}(\mu)) = \\ & \alpha(\lambda, \mu)(\tilde{Y}_k(\mu)Y_{l-2}(\mu))(\tilde{Y}_{k+1}(\lambda)X_{l-1}(\lambda)) - \beta_{l-1}(\lambda, \mu)(\tilde{Y}_k(\lambda)Y_{l-2}(\lambda))(\tilde{Y}_k(\mu)X_{l-1}(\mu)) \end{aligned} \quad (\text{B.53})$$

Appendix C. The string operator.

The string operator (5) found in [14] is written in terms of the quantities

$$\hat{Z}_1(\lambda_c) = \frac{\hat{X}(\lambda_c)}{\hat{Y}(\lambda_c)} \quad (\text{C.1})$$

with

$$\hat{X}(\lambda_c) = -2 \sum_{k=0}^{L_s-1} k \frac{\omega^{-2(k+1)}\rho_{k+1}}{P_k P_{k+1}} \quad (\text{C.2})$$

$$\hat{Y}(\lambda_c) = \sum_{k=0}^{L_s-1} \frac{\omega^{-2(k+1)}\rho_{k+1}}{P_k P_{k+1}} \quad (\text{C.3})$$

and

$$\hat{Z}_j(\lambda_c) = \hat{Z}_1(\lambda_c - (j-1)2\eta) \quad (\text{C.4})$$

where $\omega = e^{2\pi im/L}$ is a L th root of unity.

$$\rho_k = h^N(\lambda_c - (2k-1)\eta) \quad P_k = \prod_{m=1}^{n_r} h(\lambda_c - \lambda_m^r - 2k\eta). \quad (\text{C.5})$$

$\lambda_m^r, m = 1, \dots, n_r$ denote regular Bethe-roots, λ_c denotes the string center.

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